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Magnetohydrodynamic flows in porous media

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The aim of this work is to investigate the tensorial filtration law in rigid porous media for steady-state slow flow of an electrically conducting, incompressible and viscous Newtonian fluid in the presence of a magnetic field. The seepage law under a magnetic field is obtained by upscaling the flow at the pore scale. The macroscopic magnetic field and electric flux are also obtained. We use the method of multiple-scale expansions which gives rigorously the macroscopic behaviour without any preconditions on the form of the macroscopic equations. For finite Hartmann number, i.e. $\varepsilon \ll Ha \ll \varepsilon^{-1}$, and finite load factor, i.e. $\varepsilon \ll \mathscr{K} \ll \varepsilon^{-1}$, where ε characterizes the separation of scales, the macroscopic mass flow and electric current are coupled and both depend on the macroscopic gradient of pressure and the electric field. The effective coefficients satisfy the Onsager relations. In particular, the filtration law is shown to resemble Darcy's law but with an additional term proportional to the electric field. The permeability tensor, which strongly depends on the magnetic induction, i.e. Hartmann number, is symmetric, positive and satisfies the filtration analogue of the Hall effect.

1. Introduction

The study of the flow of an electrically conducting fluid through a porous medium in the presence of a magnetic field spans a range of scientific and engineering domains, including earth science, nuclear engineering and metallurgy. In nuclear engineering, knowledge of the MHD (magnetohydrodynamic) flow in a porous medium is required for the design of a blanket of liquid metal around a thermonuclear fusion–fission hybrid reactor (McWhirter *et al.* 1998*b*). In metallurgy, a permanent magnetic field can be applied during the solidification process to modify the intensity of the interdendritic flow of the metallic liquid in the mushy zone, i.e. a porous medium. This technique allows the reduction of micro–macrosegregation occurring during casting processes (see Prescott & Incropera 1993, 1996, Lehmann *et al.* 1998*a, b*).

The common characteristic of these MHD flows in porous media is that they are all electromagnetically braked by the Lorentz force

$$\boldsymbol{F}_L = \boldsymbol{J} \times \boldsymbol{B}, \tag{1.1}$$

where $\boldsymbol{B}$ is the magnetic induction applied. $\boldsymbol{J}$ is the electric current density which is related to the fluid velocity $\boldsymbol{v}$ by Ohm's law (Moreau 1990):

$$\boldsymbol{J} = \sigma_f(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

where σ_f is the electrical conductivity of the fluid and $\boldsymbol{E}$ is the electric field. Thus, the

Lorentz force (1.1) takes the form

$$\boldsymbol{F}_L = \sigma_f \boldsymbol{E} \times \boldsymbol{B} + \sigma_f (\boldsymbol{v} \times \boldsymbol{B}) \times \boldsymbol{B}.$$

The last term of this relationship represents a braking force per unit of volume due to the magnetic field. The influence of this braking force by comparison to the viscous friction is usually measured by the Hartmann number

$$Ha = \left(\frac{\sigma_f}{\mu}\right)^{1/2} Bh,$$

where μ is the fluid viscosity and h is a characteristic length of the flow.

One can find in the literature few experimental and theoretical investigations on magnetic effects on the filtration law in porous media. The first experimental study was carried out by Wallace, Pierce & Swayer (1969). In this work, the authors proposed using magnetic fields to provide a technique for studying pore size distribution in a porous medium. Experiments on the flow of mercury in porous media (sandstone) either with no magnetic field, or with a transverse magnetic field and in presence of crosswise electric currents were performed. The authors observed no change of the flow rate of mercury through the porous media when a transverse magnetic field is applied alone. This result may be due to the combination of low magnetic field and small characteristic pore length, which gives a small Hartmann number. However, they observed a change of the flow rate of mercury when a transverse magnetic field and an electric current were applied simultaneously. In order to validate these experimental observations, Rudraiah, Ramaiah & Rajasekhar (1975) carried out a theoretical and numerical study of Hartmann flow over a non-conducting permeable bed. In that work, magnetic effects are considered in the filtration law in a phenomenological way by the direct introduction in the isotropic Darcy's law (Darcy 1856) of a magnetic term proportional to the Lorentz force (1.1),

$$\boldsymbol{q} = \frac{k}{\mu}(-\nabla p + \boldsymbol{J} \times \boldsymbol{B}), \tag{1.2}$$

where $\boldsymbol{q}$ is the flow rate vector, k is the permeability, ∇p is the fluid pressure gradient. Rudraiah *et al.* (1975) found that the volume rate of flow through porous media decreases considerably on increasing the magnetic field. Under particular conditions ($B = 0,25\,\mathrm{T}$, $\sigma_f = 10^6\,\Omega^{-1}\,\mathrm{m}^{-1}$, $\mu = 1.6 \times 10^{-2}\,\mathrm{Pa\,s}$, $k = 5 \times 10^{-6}\,\mathrm{m}^2$, $Ha = 8.8$), they found about 92% fractional decrease in the volume rate of flow of mercury due to the magnetic field. More recently, an analytical model for inertialess magnetohydrodynamic flow in packed beds has been developed by McWhirter *et al.* (1998*b*). The effect of the conductivity of the porous medium is accounted for by an analogy with Hartmann flow in a cylindrical duct under an external load. The predictions of this analytical model were then compared with experimental data on the flow of an eutectic mixture of sodium and potassium NaK ($\sigma_f = 6 \times 10^6\,\Omega^{-1}\,\mathrm{m}^{-1}$) through packed beds of stainless steel spheres (McWhirter, Crawford & Klein 1998*a*). Good agreement was found between data for a wide range of Hartmann numbers ($Ha < 250$).

Also, several analytical and numerical works in the literature are devoted to the study of the magnetohydrodynamic flow of a conducting fluid through a porous medium between two parallel fixed plates (Ram & Mishra 1977; Tawil & Kamel 1994), or bounded by an infinite vertical plate (Raptis & Perdikis 1985; Yih 1998), or in a circular pipe (Ram & Mishra 1977) or during the solidification process (Prescott & Incropera 1993). The expression most often used in these works to describe the flow in the porous medium is

$$\rho \frac{\partial \langle \boldsymbol{v} \rangle}{\partial t} = -\nabla p + \mu \nabla^2 \langle \boldsymbol{v} \rangle - \mu \frac{\langle \boldsymbol{v} \rangle}{k} + \boldsymbol{J} \times \boldsymbol{B}, \tag{1.3}$$

where $\langle \boldsymbol{v} \rangle$ is the mean fluid velocity in the porous medium, which is equivalent to $\boldsymbol{q}$ in (1.2). The fluid flow also satisfies the zero-divergence condition. Equation (1.3) reduces to the modified Darcy's law (1.2) for low values of the permeability k and to the Navier–Stokes equation with magnetic effects for high values of k. The transition between these two regimes occurs when the Brinkman term $\mu \nabla^2 \langle \boldsymbol{v} \rangle$ is of the same order of magnitude as $\mu \langle \boldsymbol{v} \rangle / k$ (Brinkman 1947). Thus, equation (1.3) can be simply viewed as a superposition of Darcy's law and the Navier–Stokes equation. It is of practical interest for numerical investigations. However, Levy (1981, 1983) has shown that Brinkman's equation is valid for very large characteristic porosity $\phi_c \approx 1 - 3\varepsilon^2$, where the scale ratio ε is defined by l/L with l and L the characteristic lengths of the heterogeneities and the macroscopic sample or excitation, respectively. The porosity ϕ is the ratio of the pore volume to the total volume. For a typical value $\varepsilon = 10^{-3}$, which yields $\phi_c \approx 0.999997$. For such values of ϕ_c, the solid is generally not connected and the fluid–solid system behaves as a suspension, which cannot be of Brinkman type. Therefore the range of validity of equation (1.3) is quite questionable.

Finally, considering a porous medium saturated by an electrolyte and neglecting the Lorentz force, Pride (1994) has derived by volume averaging the macroscopic description for the coupled electromagnetics and acoustics. This consists of two coupled Maxwell and Biot equations (Biot 1956). Due to the small Lorentz force, effective coefficients do not depend on the magnetic field and the coupling between mass and electric fluxes is shown to be symmetrical in the case of isotropic media.

The aim of this work is to investigate the tensorial filtration law in rigid porous media for steady-state slow flow of an electrically conducting, incompressible and viscous Newtonian fluid in the presence of a magnetic field. We have in mind applications such as MHD processes in metallurgy, where the Lorentz force cannot be neglected. We use an upscaling technique, i.e. the method of multiple scale expansions, to determine the macroscopic flow from the description of the physical mechanisms at the pore scale. This upscaling technique allows the equivalent macroscopic behaviour of a heterogeneous system, as for example a porous medium, to be derived if the condition of separation of scales is satisfied (Bensoussan, Lions & Papanicolaou 1978; Sanchez-Palencia 1980; Auriault 1991):

$$\varepsilon = \frac{l}{L} \ll 1, \tag{1.4}$$

where l and L are the characteristic lengths of the heterogeneities and the macroscopic sample or excitation, respectively. Under these conditions, the corresponding macroscopic descriptions are intrinsic to the geometry of the medium and the phenomenon. They are also independent of the macroscopic boundary conditions. In this study, we follow the approach suggested in Auriault (1991). The macroscopic equivalent model is obtained from the description at the heterogeneity scale by: (i) assuming the medium to be periodic, without loss of generality since (1.4) is fulfilled; (ii) writing the local description in a dimensionless form; (iii) evaluating the dimensionless numbers with respect to the scale ratio ε; (iv) looking for the unknown fields in the form of asymptotic expansions in powers of ε; (v) solving the successive boundary-value problems that are obtained after introducing these expansions in the local dimensionless description. The macroscopic equivalent model is obtained from compatibility

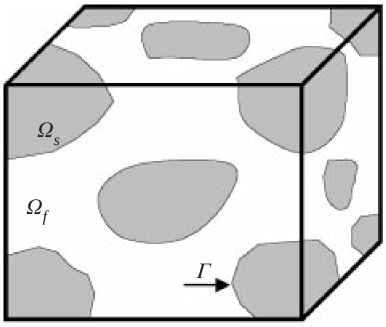

FIGURE 1. Representative elementary volume or periodic cell Ω.

conditions which are the necessary conditions for the existence of solutions to the boundary-value problems.

The description at the pore scale of the magnetohydrodynamic flow of a conducting fluid in a rigid porous medium is given in § 2. This description is made dimensionless and we evaluate the different dimensionless numbers with respect to the scale ratio ε. Section 3 is devoted to the upscaling for steady-state flow with finite Hartmann number, $\varepsilon \ll Ha \ll \varepsilon^{-1}$. The macroscopic mass flow and electric current are coupled and both depend on the macroscopic gradient of pressure and electric field. The seepage law under the magnetic field is shown to resemble Darcy's law but with an additional term proportional to the electric field. The properties of the effective coefficients are investigated in § 4. It is shown that the effective coefficients which strongly depend on the magnetic induction, i.e. the Hartmann number, satisfy Onsager relations. In particular, it is shown that the permeability tensor is symmetric, positive and satisfies the filtration analogue of the Hall relation. However, unlike Onsager approach, the symmetry of the coupling is obtained here from the local field properties by using a deterministic approach. Finally, an example based on the well-known Hartmann problem (Hartmann 1937) is presented in § 5.

2. Local flow description and estimations

2.1. *Local flow description*

Consider the flow of an electrically conducting, incompressible and viscous Newtonian fluid through a rigid conducting porous medium in the presence of a magnetic field. The porous medium is spatially periodic and consists of repeated unit cells (parallelepipeds). A period is shown in figure 1. There are two characteristic length scales in this problem: the characteristic microscopic length scale l of the pores and of the unit cell, and the macroscopic length scale that may be represented by either the macroscopic pressure drop scale or by the sample size scale. For simplicity, we assume both macroscopic length scales to be of similar order of magnitude, $O(L)$. Moreover we assume that the two length scales l and L are well separated: $l \ll L$. The unit cell is denoted by Ω and is bounded by $\partial\Omega$, the fluid part of the unit cell is denoted by Ω_f, the solid part of the unit cell is denoted by Ω_s, and the fluid–solid interface inside the unit cell is Γ.

In this study, we assume a steady-state slow flow. Consequently, in the pores, the governing equations for the flow of an electrically conducting and incompressible

Newtonian fluid with invariant physical properties are (Moreau 1990)

$$-\nabla p + \mu \nabla^2 \boldsymbol{v} + \boldsymbol{J} \times \boldsymbol{B} = \mathbf{0} \quad \text{in } \Omega_f, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega_f, \tag{2.2}$$

where $\boldsymbol{v}$ is the velocity, p is the pressure, μ is the viscosity and $\boldsymbol{B}$ is the magnetic induction in the fluid. The current density $\boldsymbol{J}$ in the fluid is related to the fluid velocity $\boldsymbol{v}$ and the electric field $\boldsymbol{E}$ by Ohm's law, and to the magnetic field $\boldsymbol{H}$ by Maxwell's relation

$$\boldsymbol{J} = \sigma_f(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \nabla \times \boldsymbol{H} \quad \text{in } \Omega_f, \tag{2.3}$$

where σ_f is the electrical conductivity. The conducting solid skeleton Ω_s is assumed to be rigid. In the solid, the above relation reduces to

$$\boldsymbol{J} = \sigma_s \boldsymbol{E} = \nabla \times \boldsymbol{H} \quad \text{in } \Omega_s. \tag{2.4}$$

From (2.3) and (2.4) we have

$$\nabla \cdot \boldsymbol{J} = 0 \quad \text{in } \Omega. \tag{2.5}$$

The following relations are also valid:

$$\nabla \times \boldsymbol{E} = \mathbf{0} \quad \text{in } \Omega, \tag{2.6}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{with} \quad \boldsymbol{B} = \mu^* \boldsymbol{H} \quad \text{in } \Omega. \tag{2.7}$$

For simplicity, we have assumed that the conductivity σ and the magnetic permeability μ^* are isotropic. In particular we do not consider the Hall effect at the microscale. Finally, the set of equations (2.1)–(2.7) is completed by the adherence condition on the solid–liquid interface Γ,

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.8}$$

and the continuity on Γ of the normal components of the magnetic induction and the current density, and of the tangential components of the magnetic field and the electric field

$$(\boldsymbol{B}_f - \boldsymbol{B}_s) \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma, \tag{2.9}$$

$$(\boldsymbol{J}_f - \boldsymbol{J}_s) \cdot \boldsymbol{N} = (\sigma_f \boldsymbol{E}_f - \sigma_s \boldsymbol{E}_s) \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma, \tag{2.10}$$

$$(\boldsymbol{H}_f - \boldsymbol{H}_s) \times \boldsymbol{N} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.11}$$

$$(\boldsymbol{E}_f - \boldsymbol{E}_s) \times \boldsymbol{N} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.12}$$

where $\boldsymbol{N}$ is the unit outward vector of Γ.

2.2. *Estimations of dimensionless numbers*

Since the ratio between microscopic and macroscopic length scales is small, the fundamental perturbation parameter ε is chosen to be

$$\varepsilon = \frac{l}{L}, \quad \varepsilon \ll 1.$$

The independent dimensionless numbers which characterize the magnetohydrodynamic liquid flow problem may be related to the magnitude of ε. We use the local length scale of a pore l as the characteristic length scale for the variations of the differential operators: we apply the so-called microscopic point of view (Auriault 1991). The local flow description introduces six dimensionless numbers: the ratio Q_l of the pressure and the viscous forces, the Hartmann number Ha_l that characterizes the ratio of the electromagnetic forces to the viscous forces, the ratio Rm_l (magnetic Reynolds number) of $\sigma_f\ \boldsymbol{v} \times \boldsymbol{B}$ to $\nabla \times \boldsymbol{H}$, the load factor $\mathscr{K}$ which is the ratio of the electric field to $\boldsymbol{v} \times \boldsymbol{B}$ and the ratios S and M of the electrical conductivities and of the magnetic permeabilities, respectively, of the fluid and the solid skeleton. For evaluating these dimensionless numbers, we introduce characteristic values (denoted with the subscript c) that are related to the physical phenomenon (pressure drop p_c, fluid velocity v_c, electric field E_c, magnetic induction B_c and length l). We obtain

$$Q_l = \frac{|\nabla p|}{|\mu \nabla^2 \boldsymbol{v}|} = \frac{p_c l}{\mu_c v_c},$$

$$Ha_l = \left(\frac{|\sigma_f\ \boldsymbol{v} \times \boldsymbol{B}|}{|\mu \nabla^2 \boldsymbol{v}|} \right)^{1/2} = B_c l \left(\frac{\sigma_{fc}}{\mu_c} \right)^{1/2},$$

$$Rm_l = \frac{|\sigma_f\ \boldsymbol{v} \times \boldsymbol{B}|}{|\nabla \times \boldsymbol{H}|} = \mu^*_{fc} \sigma_{fc} v_c l,$$

$$\mathscr{K} = \frac{|\boldsymbol{E}|}{|\boldsymbol{v} \times \boldsymbol{B}|} = \frac{E_c}{v_c B_c},$$

$$S = \frac{\sigma_{fc}}{\sigma_{sc}}, \quad M = \frac{\mu^*_{fc}}{\mu^*_{sc}}.$$

It can be shown by simple physical reasoning (Auriault 1991) that the problem is homogenizable if $Q_l = O(\varepsilon^{-1})$. The other dimensionless numbers depend on the problem under consideration. For example, in a metallic mushy zone (Moreau 1990; Lehmann *et al.* 1998*a*) $\sigma_{sc} \approx \sigma_{fc} \approx 10^6\ \Omega^{-1}\,\mathrm{m}^{-1}$, $\mu^*_{fc} \approx \mu^*_{sc}$ and $\mu^*_c \sigma_c \approx 1\,\mathrm{m}^{-2}\,\mathrm{s}$, $\mu_c \approx 10^{-3}\,\mathrm{Pa\,s}$, $B_c \approx 1\,\mathrm{T}$, $E_c \approx 10^{-2}\,\mathrm{V\,m}^{-1}$, $l \approx 100\,\mu\mathrm{m}$, $L \approx 10\,\mathrm{cm}$, $v_c \approx 10^{-3}\,\mathrm{m\,s}^{-1}$. Thus we obtain $\varepsilon \approx 10^{-3}$ and $Ha_l = O(1)$, $Rm_l = O(\varepsilon^2)$, $\mathscr{K} = O(1)$, $S = O(1)$ and $M = O(1)$. *For simplicity we use similar notation for dimensionless and dimensional quantities.* The microscopic dimensionless set of equations that describes the magnetohydrodynamic flow in a porous medium is the following, in which all quantities (p, $\boldsymbol{v}$, $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{E}$, $\boldsymbol{J}$, μ, μ^*, σ) are now dimensionless ($O(1)$):

$$\mu \nabla^2 \boldsymbol{v} - \varepsilon^{-1} \nabla p + \sigma_f (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{0} \quad \text{in } \Omega_f, \tag{2.13}$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega_f, \tag{2.14}$$

and we have

$$\varepsilon^2 \sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \nabla \times \boldsymbol{H} \quad \text{in } \Omega, \tag{2.15}$$

$$\nabla \cdot \boldsymbol{J} = \nabla \cdot [\sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})] = 0 \quad \text{in } \Omega, \tag{2.16}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \boldsymbol{B} = \mu^* \boldsymbol{H} \quad \text{in } \Omega, \tag{2.17}$$

$$\nabla \times \boldsymbol{E} = \boldsymbol{0} \quad \text{in } \Omega, \tag{2.18}$$

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{2.19}$$

$$(\boldsymbol{B}_f - \boldsymbol{B}_s) \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma, \tag{2.20}$$

$$(\boldsymbol{J}_f - \boldsymbol{J}_s) \cdot \boldsymbol{N} = (\sigma_f \boldsymbol{E}_f - \sigma_s \boldsymbol{E}_s) \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma, \tag{2.21}$$

$$(\boldsymbol{H}_f - \boldsymbol{H}_s) \times \boldsymbol{N} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{2.22}$$

$$(\boldsymbol{E}_f - \boldsymbol{E}_s) \times \boldsymbol{N} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{2.23}$$

where $\boldsymbol{N}$ is a unit normal to Γ. In the above equations, the velocity $\boldsymbol{v}$ cancels out in Ω_s. Note that the dimensionless viscosity $\mu = 1$ is maintained in equation (2.13) for notation consistency. Equation (2.15) shows that up to the second order of approximation, $\nabla \times \boldsymbol{H}$ is null and $\boldsymbol{H}$ is the gradient of a potential ψ which can be chosen as it is continuous everywhere. We seek a macroscopic description, at the scale L. Therefore we have the following estimate

$$H_c = O\left(\frac{\psi_c}{L}\right) = O\left(\frac{\varepsilon^{-1}\psi_c}{l}\right).$$

When using l to normalize, we obtain in the dimensionless form

$$\boldsymbol{H} = -\varepsilon^{-1}\nabla\psi + O(\varepsilon^2). \tag{2.24}$$

Similarly, equation (2.18) shows that the electric field $\boldsymbol{E}$ derives from a potential V. By following the same route as for $\boldsymbol{H}$, this is written in the following dimensionless form:

$$\boldsymbol{E} = -\varepsilon^{-1}\nabla V. \tag{2.25}$$

The orders of magnitude of the Hartmann number Ha and the load factor $\mathscr{K}$ ensure the coupling of the velocity field and of the electric flux at the pore scale. Therefore, we can anticipate that the macroscopic corresponding fluxes, which are averages of these local fluxes, are also coupled.

3. Homogenization

The next step is to introduce multiple-scale coordinates (Bensoussan *et al.* 1978; Sanchez-Palencia 1980). The two characteristic lengths L and l introduce two dimensionless space variables,

$$\boldsymbol{x} = \frac{\boldsymbol{X}}{L}, \quad \boldsymbol{y} = \frac{\boldsymbol{X}}{l},$$

where $\boldsymbol{X}$ is the physical space variable. The macroscopic space variable $\boldsymbol{x}$ is related to the microscopic space variable $\boldsymbol{y}$ by $\boldsymbol{x} = \varepsilon\boldsymbol{y}$. When using l as the characteristic length the dimensionless derivative operator becomes

$$\nabla = \nabla_y + \varepsilon\nabla_x,$$

where the subscripts x and y denote the derivatives with respect to the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. Following the multiple-scale expansion technique (Bensoussan *et al.* 1978; Sanchez-Palencia 1980), the velocity $\boldsymbol{v}$, the pressure fluctuation p, the electric field $\boldsymbol{E}$, the electric potential V, the magnetic flux density $\boldsymbol{B}$, the magnetic field $\boldsymbol{H}$ and the magnetic potential ψ are sought in the form of asymptotic expansions of powers of ε

$$\varphi = \varphi^{(0)}(\boldsymbol{x},\boldsymbol{y}) + \varepsilon\, \varphi^{(1)}(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2\, \varphi^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \cdots,$$

where $\varphi = \boldsymbol{v}, p, \boldsymbol{E}, V, \boldsymbol{B}, \boldsymbol{H}, \psi$ and the corresponding $\varphi^{(i)}$ are periodic functions or vectors of period Ω with respect to space variable $\boldsymbol{y}$. Substituting these expansions in the set (2.13)–(2.25) gives, by identification of like powers of ε, successive boundary value problems to be investigated.

The magnetic Reynolds number Rm is very small, therefore we first address the macroscopic description of the magnetic field which is independent of the mass flow and the electric current.

3.1. *Macroscopic magnetic field and induction*

Introducing asymptotic expansions for $\boldsymbol{H}$ and ψ in the relation (2.24) gives at the lowest order

$$\nabla_y \psi^{(0)} = \mathbf{0}, \quad \psi^{(0)} = \psi^{(0)}(\boldsymbol{x}).$$

From (2.24), we have at the next order

$$\boldsymbol{H}^{(0)} = -\nabla_y \psi^{(1)} - \nabla_x \psi^{(0)}, \tag{3.1}$$

where $\nabla_x \psi^{(0)}$ is considered as being given. Then $\psi^{(1)}$ is given by the following boundary value problem which is obtain from (2.17), (2.20) and (3.1):

$$\nabla_y \cdot [\mu^* (\nabla_y \psi^{(1)} + \nabla_x \psi^{(0)})] = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$[\mu_f^* (\nabla_y \psi_f^{(1)} + \nabla_x \psi^{(0)}) - \mu_s^* (\nabla_y \psi_s^{(1)} + \nabla_x \psi^{(0)})] \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma, \tag{3.3}$$

where $\psi^{(1)}(\boldsymbol{x}, \boldsymbol{y})$ is Ω-periodic and continuous on Γ. The potential $\psi^{(1)}$ appears as a linear function of $\nabla_x \psi^{(0)}$, to an added arbitrary function of $\boldsymbol{x}$:

$$\psi^{(1)}(\boldsymbol{x}, \boldsymbol{y}) = -\boldsymbol{m}(\boldsymbol{y}) \cdot \nabla_x \psi^{(0)} + \bar{\psi}^{(1)}(\boldsymbol{x}), \tag{3.4}$$

where $m_i(\boldsymbol{y})$, of zero average, is the solution of the boundary value problem (3.2)–(3.3) for $\partial \psi^{(0)}/\partial x_j = -\delta_{ji}$, where δ_{ij} is the Kronecker symbol. Note that

$$\langle \boldsymbol{H}^{(0)} \rangle = \langle -\nabla_y \psi^{(1)} - \nabla_x \psi^{(0)} \rangle = -\nabla_x \psi^{(0)}, \tag{3.5}$$

$$\nabla_x \times \langle \boldsymbol{H}^{(0)} \rangle = \mathbf{0}. \tag{3.6}$$

Then (2.17) and (2.20) at the order ε yield

$$\nabla_y \cdot \boldsymbol{B}^{(1)} + \nabla_x \cdot \boldsymbol{B}^{(0)} = 0 \quad \text{in } \Omega, \tag{3.7}$$

$$(\boldsymbol{B}_f^{(1)} - \boldsymbol{B}_s^{(1)}) \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma. \tag{3.8}$$

Integrating (3.7) over Ω, using the divergence theorem, relation (3.8) and the periodicity gives the macroscopic model for the magnetic field:

$$\nabla_x \cdot \langle \boldsymbol{B}^{(0)} \rangle = 0, \quad \langle \boldsymbol{B}^{(0)} \rangle = \boldsymbol{\mu}^{*\text{eff}} \langle \boldsymbol{H}^{(0)} \rangle = -\boldsymbol{\mu}^{*\text{eff}} \nabla_x \psi^{(0)}, \tag{3.9}$$

where $\boldsymbol{\mu}^{*\text{eff}}$ is the effective magnetic permeability tensor defined as

$$\mu_{ij}^{*\text{eff}} = \left\langle \mu^* \left(I_{ij} - \frac{\partial m_j}{\partial y_i} \right) \right\rangle,$$

where $\boldsymbol{I}$ is the identity tensor. We have

$$B_i^{(0)}(\boldsymbol{x}, \boldsymbol{y}) = -\mu^* \left(I_{ij} - \frac{\partial m_j}{\partial y_i} \right) \frac{\partial \psi^{(0)}}{\partial x_j}. \tag{3.10}$$

A similar result could be obtained by following the method shown in Sanchez-Palencia (1974).

3.2. *Macroscopic mass and electric fluxes*

Relation (2.25) gives at the lowest order

$$\nabla_y V^{(0)} = \mathbf{0}, \quad V^{(0)} = V^{(0)}(\boldsymbol{x}),$$

and we have

$$\langle \boldsymbol{E}^{(0)} \rangle = \langle -\nabla_y V^{(1)} - \nabla_x V^{(0)} \rangle = -\nabla_x V^{(0)}, \tag{3.11}$$

$$\nabla_x \times \langle \boldsymbol{E}^{(0)} \rangle = \mathbf{0}. \tag{3.12}$$

From (2.13) at the order ε^{-1}, the lowest-order approximation of the pressure satisfies

$$\nabla_y p^{(0)} = \mathbf{0}, \quad p^{(0)} = p^{(0)}(\boldsymbol{x}).$$

The potential $V^{(1)}$, the first-order approximation of the velocity $\boldsymbol{v}^{(0)}$ and the second-order approximation of the pressure $p^{(1)}$ are determined by the following set which is obtained from (2.13), (2.14), (2.16), (2.19), (2.21) and (2.25) at the order ε^0:

$$\mu\nabla_y^2\boldsymbol{v}^{(0)} - \nabla_x p^{(0)} - \nabla_y p^{(1)} = -\sigma_f(-\nabla_y V_f^{(1)} - \nabla_x V^{(0)} + \boldsymbol{v}^{(0)} \times \boldsymbol{B}^{(0)}) \times \boldsymbol{B}^{(0)} \quad \text{in } \Omega_f, \tag{3.13}$$

$$\nabla_y \cdot \boldsymbol{v}^{(0)} = 0 \quad \text{in } \Omega_f, \tag{3.14}$$

$$\nabla_y \cdot [\sigma(-\nabla_y V^{(1)} - \nabla_x V^{(0)} + \boldsymbol{v}^{(0)} \times \boldsymbol{B}^{(0)})] = 0 \quad \text{in } \Omega, \tag{3.15}$$

$$\boldsymbol{v}^{(0)} = \mathbf{0} \quad \text{on } \Gamma, \tag{3.16}$$

$$(\sigma_f(\nabla_y V_f^{(1)} + \nabla_x V^{(0)}) - \sigma_s(\nabla_y V_s^{(1)} + \nabla_x V^{(0)})) \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma, \tag{3.17}$$

where $V^{(1)}$, $\boldsymbol{v}^{(0)}$ and $p^{(1)}$ are Ω-periodic. In equation (3.15), $\boldsymbol{v}^{(0)}$ cancels out in Ω_s. $\boldsymbol{B}^{(0)}$ is given by (3.10). Vector $\nabla_x p^{(0)}$ is the macroscopic driving pressure force, whereas $\nabla_x V^{(0)}$ is the macroscopic gradient electric potential.

The above set of partial differential equations will be used in § 5 to determine the solutions in parallel plane fissures. However, we need an equivalent variational formulation to study the existence and uniqueness of the solution and to investigate in § 4 the properties of the effective macroscopic coefficients. Let us introduce two Hilbert spaces. Space $\mathscr{V}$ is the space of Ω-periodic functions α defined in Ω that are of zero average over Ω and that have the scalar product, for $\alpha, \beta \in \mathscr{V}$,

$$(\alpha, \beta)_{\mathscr{V}} = \int_\Omega \sigma \frac{\partial \alpha}{\partial y_i}\frac{\partial \beta}{\partial y_i}\,\mathrm{d}\Omega.$$

Throughout the paper $\mathrm{d}\Omega$ denotes an integration with respect to space variable $\boldsymbol{y}$: $\mathrm{d}\Omega = \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3$. The space $\mathscr{W}$ is the space of Ω-periodic, divergence-free vectors $\boldsymbol{u}$, defined in Ω_f, where the vectors vanish on Γ, and with the scalar product, for $\boldsymbol{u}$, $\boldsymbol{v} \in \mathscr{W}$,

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathscr{W}} = \int_{\Omega_f} \mu \frac{\partial u_i}{\partial y_j}\frac{\partial v_i}{\partial y_j}\,\mathrm{d}\Omega.$$

Now, let us multiply equations (3.13) and (3.15) by $\boldsymbol{u} \in \mathscr{W}$ and $\alpha \in \mathscr{V}$ and integrate over Ω_f and Ω, respectively. By using integration by parts, the divergence theorem, periodicity, and the boundary conditions on Γ, one obtains the equivalent variational formulation:

$$\forall \alpha \in \mathscr{V}, \quad (V^{(1)}, \alpha)_{\mathscr{V}} = \int_\Omega \sigma(-\nabla_x V^{(0)} + \boldsymbol{v}^{(0)} \times \boldsymbol{B}^{(0)}) \cdot \nabla_y \alpha\,\mathrm{d}\Omega, \tag{3.18}$$

$$\begin{aligned}\forall \boldsymbol{u} \in \mathscr{W}, \quad (\boldsymbol{v}^{(0)}, \boldsymbol{u})_{\mathscr{W}} = &-\int_{\Omega_f} \boldsymbol{u} \cdot \nabla_x p^{(0)}\,\mathrm{d}\Omega + \int_{\Omega_f} \sigma_f\ [(\boldsymbol{v}^{(0)} \times \boldsymbol{B}^{(0)}) \times \boldsymbol{B}^{(0)}] \cdot \boldsymbol{u}\,\mathrm{d}\Omega \\ &+ \int_{\Omega_f} \sigma_f[(-\nabla_y V^{(1)} - \nabla_x V^{(0)}) \times \boldsymbol{B}^{(0)}] \cdot \boldsymbol{u}\,\mathrm{d}\Omega. \end{aligned} \tag{3.19}$$

In (3.18), $\boldsymbol{v}^{(0)}$ cancels out in Ω_s. Formulations (3.18) and (3.19) and Lax–Milgram lemma ensure a unique solution for $V^{(1)}$ and $\boldsymbol{v}^{(0)}$ in $\mathscr{V}$ and $\mathscr{W}$, respectively. Due to the linearity of (3.18) and (3.19) with respect to $\nabla_x V^{(0)}$ and $\nabla_x p^{(0)}$, $V^{(1)}$ and $\boldsymbol{v}^{(0)}$ are linear functions and vectors, respectively, of these quantities, with an added arbitrary function of $\boldsymbol{x}$ to $V^{(1)}$. They can be put into the form

$$V^{(1)} = \boldsymbol{\tau} \cdot \nabla_x V^{(0)} + \boldsymbol{\eta} \cdot \nabla_x p^{(0)} + \bar{V}^{(1)}(\boldsymbol{x}), \quad \boldsymbol{v}^{(0)} = -\boldsymbol{k}\nabla_x p^{(0)} - \boldsymbol{\chi}\nabla_x V^{(0)}, \tag{3.20}$$

where the vectors $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$, and the tensors $\boldsymbol{k}$ and $\boldsymbol{\chi}$ depend on $\langle \boldsymbol{H}^{(0)} \rangle = -\nabla_x \psi^{(0)}$ and $\boldsymbol{y}$. Therefore, from (3.20) and (3.13), the fluid pressure $p^{(1)}$ also appears as a linear function of $\nabla_x V^{(0)}$ and $\nabla_x p^{(0)}$ with an added arbitrary function of $\boldsymbol{x}$,

$$p^{(1)} = -\boldsymbol{\zeta} \cdot \nabla_x V^{(0)} - \boldsymbol{\xi} \cdot \nabla_x p^{(0)} + \bar{p}^{(1)}(\boldsymbol{x}), \tag{3.21}$$

where the two vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ depend on $\langle \boldsymbol{H}^{(0)} \rangle = -\nabla_x \psi^{(0)}$ and $\boldsymbol{y}$. Note that $-\tau_j$, $-\eta_j$, ζ_j, ξ_j, k_{ij} and χ_{ij} are solutions of the boundary value problem (3.13)–(3.17) for $\partial p^{(0)}/\partial x_i = -\delta_{ij}$ and $\partial V^{(0)}/\partial x_i = -\delta_{ij}$, where δ_{ij} is the Kronecker delta.

Finally, the macroscopic description is obtained from balances (2.14) and (2.16) and continuity conditions (2.19) and (2.21) at the order ε,

$$\nabla_y \cdot \boldsymbol{v}^{(1)} + \nabla_x \cdot \boldsymbol{v}^{(0)} = 0 \quad \text{in } \Omega_f, \tag{3.22}$$

$$\nabla_y \cdot \boldsymbol{J}^{(1)} + \nabla_x \cdot \boldsymbol{J}^{(0)} = 0 \quad \text{in } \Omega, \tag{3.23}$$

$$\boldsymbol{v}^{(1)} = \boldsymbol{0}, \quad (\boldsymbol{J}_f^{(1)} - \boldsymbol{J}_s^{(1)}) \cdot \boldsymbol{N} = 0 \quad \text{on } \Gamma. \tag{3.24}$$

Integrating relations (3.22) and (3.23) in their respective domains of definition, using the divergence operator, the periodicity and continuity conditions (3.24) yields

$$\nabla_x \cdot \langle \boldsymbol{v}^{(0)} \rangle = 0, \quad \langle \boldsymbol{v}^{(0)} \rangle = -\boldsymbol{K}\nabla_x p^{(0)} - \boldsymbol{\chi}^{\text{eff}}\nabla_x V^{(0)}, \tag{3.25}$$

$$K_{ij} = \langle k_{ij} \rangle, \quad \chi_{ij}^{\text{eff}} = \langle \chi_{ij} \rangle, \tag{3.26}$$

and

$$\nabla_x \cdot \langle \boldsymbol{J}^{(0)} \rangle = 0, \quad \langle \boldsymbol{J}^{(0)} \rangle = -\boldsymbol{\sigma}^{\text{eff}}\nabla_x V^{(0)} - \boldsymbol{\eta}^{\text{eff}}\nabla_x p^{(0)}, \tag{3.27}$$

$$\sigma_{ij}^{\text{eff}} = \left\langle \sigma\left(\frac{\partial \tau_j}{\partial y_i} + I_{ij}\right) + \sigma\epsilon_{ikl}\,\chi_{kj}\,B_l^{(0)} \right\rangle, \quad \eta_{ij}^{\text{eff}} = \left\langle \sigma\frac{\partial \eta_j}{\partial y_i} + \sigma\epsilon_{ikl}\,k_{kj}\,B_l^{(0)} \right\rangle, \tag{3.28}$$

where ϵ_{ikl} is the permutation symbol and $\boldsymbol{I}$ is the identity tensor. In the flow laws (3.25) and (3.27), $\boldsymbol{K}$ is the permeability, $\boldsymbol{\sigma}^{\text{eff}}$ is the effective electric conductivity, $\boldsymbol{\chi}^{\text{eff}}$ is the electro-osmotic conductivity and $\boldsymbol{\eta}^{\text{eff}}$ is the electric conductivity associated to the streaming potential effect.

Note the following points:

(i) The filtration tensor $\boldsymbol{K}$ and the other effective coefficients in (3.25) and (3.27) depend on the magnetic flux $\langle \boldsymbol{B}^{(0)} \rangle = \boldsymbol{\mu}^{*\text{eff}} \langle \boldsymbol{H}^{(0)} \rangle$.

(ii) The macroscopic model given by (3.6), (3.9), (3.12), (3.25) and (3.27) is an approximation since it relates to first orders of magnitude of macroscopic physical quantities.

(iii) The macroscopic model is valid, concerning the Hartmann number, in the range $\varepsilon \ll Ha \ll \varepsilon^{-1}$ which can be quite large if the separation of scales is good.

(iv) From Levy (1981, 1983), it can be seen that the macroscopic model is valid at least in the range $\phi \leqslant 1 - 3\varepsilon$. In the present analysis that yields $\phi \leqslant 0.997$, which covers the range of practical interest.

(v) In some cases, the magnetic permeabilities in the fluid and the solid are nearly the same. When assuming $\mu_f^* = \mu_s^*$, we obtain $\boldsymbol{m} = \boldsymbol{0}$ and $\boldsymbol{B}^{(0)} = \boldsymbol{B}^{(0)}(\boldsymbol{x})$ is a constant over the period.

4. Properties of the effective coefficients

Let us first introduce some relations which will be used throughout this section. By introducing $\boldsymbol{J}^{(0)} = \sigma(-\nabla_y V^{(1)} - \nabla_x V^{(0)} + \boldsymbol{v}^{(0)} \times \boldsymbol{B}^{(0)})$ in formulations (3.18) and (3.19), we obtain successively

$$\frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot \nabla_y \alpha \, \mathrm{d}\Omega = 0, \tag{4.1}$$

$$\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{u})_{\mathscr{W}} - \frac{1}{\Omega}\int_{\Omega_f} (\boldsymbol{J}^{(0)} \times \boldsymbol{B}^{(0)}) \cdot \boldsymbol{u} \, \mathrm{d}\Omega = -\frac{1}{\Omega}\int_{\Omega_f} \boldsymbol{u} \cdot \nabla_x p^{(0)} \, \mathrm{d}\Omega. \tag{4.2}$$

Subtracting now term by term these two equations yields

$$\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{u})_{\mathscr{W}} + \frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot (-\nabla_y \alpha + \boldsymbol{u} \times \boldsymbol{B}^{(0)}) \, \mathrm{d}\Omega = -\frac{1}{\Omega}\int_{\Omega_f} \boldsymbol{u} \cdot \nabla_x p^{(0)} \, \mathrm{d}\Omega. \tag{4.3}$$

4.1. *Properties of the permeability tensor* $\boldsymbol{K}$

To investigate the properties of tensor $\boldsymbol{K}$ we now put $\nabla_x V^{(0)} = \boldsymbol{0}$ in relation (4.3).

4.1.1. $\boldsymbol{K}$ *positive*

We first study the positivity of $\boldsymbol{K}$. By letting $\alpha = V^{(1)} \neq 0$ and $\boldsymbol{u} = \boldsymbol{v}^{(0)} \neq \boldsymbol{0}$ in formulation (4.3), we obtain

$$\begin{aligned}\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{v}^{(0)})_{\mathscr{W}} + \frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot (-\nabla_y V^{(1)} + \boldsymbol{v}^{(0)} \times \boldsymbol{B}^{(0)}) \, \mathrm{d}\Omega &= -\frac{1}{\Omega}\int_{\Omega_f} \boldsymbol{v}^{(0)} \cdot \nabla_x p^{(0)} \, \mathrm{d}\Omega \\ &= \boldsymbol{K}\nabla_x p^{(0)} \cdot \nabla_x p^{(0)},\end{aligned}$$

where

$$\frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot (-\nabla_y V^{(1)} + \boldsymbol{v}^{(0)} \times \boldsymbol{B}^{(0)}) \, \mathrm{d}\Omega = \frac{1}{\Omega}\int_{\Omega} \frac{(\boldsymbol{J}^{(0)})^2}{\sigma} \, \mathrm{d}\Omega > 0$$

is the energy dissipated into heat by the Joule effect. The above inequality implies that

$$\boldsymbol{K}\nabla_x p^{(0)} \cdot \nabla_x p^{(0)} > \frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{v}^{(0)})_{\mathscr{W}} > 0,$$

which shows that tensor $\boldsymbol{K}$ is positive.

4.1.2. $\boldsymbol{K}$ *symmetric*

We now investigate the symmetries of $\boldsymbol{K}$. Quantities $v_i^{(0)} = k_{ip}$ and $V^{(1)} = -\eta_p$ are the solution of (3.18) and (3.19) for $\nabla_x V^{(0)} = \boldsymbol{0}$ and $\partial p^{(0)}/\partial x_i = -\delta_{ip}$, where δ_{ip} is the Kronecker delta. Consider formulation (4.3) successively with $v_i^{(0)} = k_{ip}$ and $v_i^{(0)} = k_{iq}$, $V^{(1)} = -\eta_p$ and $V^{(1)} = -\eta_q$, $u_i = k_{iq}$ and $u_i = k_{ip}$, $\alpha = -\eta_q$ and $\alpha = -\eta_p$, respectively. Thus, we obtain

$$\begin{aligned}(k_{ip}, k_{iq})_{\mathscr{W}} + \int_{\Omega} \sigma \left(\frac{\partial \eta_p}{\partial y_i} + \epsilon_{ijk}\, k_{jp}\, B_k^{(0)}\right)\left(\frac{\partial \eta_q}{\partial y_i} + \epsilon_{ijk}\, k_{jq}\, B_k^{(0)}\right) \mathrm{d}\Omega \\ = \int_{\Omega_f} k_{pq} \, \mathrm{d}\Omega = \Omega K_{pq},\end{aligned}$$

$$\begin{aligned}(k_{ip}, k_{iq})_{\mathscr{W}} + \int_{\Omega} \sigma \left(\frac{\partial \eta_q}{\partial y_i} + \epsilon_{ijk}\, k_{jq}\, B_k^{(0)}\right)\left(\frac{\partial \eta_p}{\partial y_i} + \epsilon_{ijk}\, k_{jp}\, B_k^{(0)}\right) \mathrm{d}\Omega \\ = \int_{\Omega_f} k_{qp} \, \mathrm{d}\Omega = \Omega K_{qp}.\end{aligned}$$

Subtracting term by term these two equations yields

$$K_{pq} = K_{qp}, \tag{4.4}$$

which shows that tensor $\boldsymbol{K}$ is symmetric. However, if Hall's effect is assumed at the pore scale, i.e. if the electric conductivity is now a non-symmetrical tensor $\boldsymbol{\sigma}$, $\sigma_{ij}(\boldsymbol{B}) \neq \sigma_{ji}(\boldsymbol{B})$ and $\sigma_{ij}(\boldsymbol{B}) = \sigma_{ji}(-\boldsymbol{B})$ (see Landau & Lifshitz 1960, p. 96), then it can be shown that the tensor $\boldsymbol{K}$ is no longer symmetric, $K_{pq} \neq K_{qp}$.

4.1.3. *Filtration analogue of Hall's effect*

By adding term by term equations (4.1) and (4.2), we now obtain

$$\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{u})_{\mathscr{W}} + \frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot (\nabla_y \alpha + \boldsymbol{u} \times \boldsymbol{B}^{(0)})\, \mathrm{d}\Omega = -\frac{1}{\Omega}\int_{\Omega_f} \boldsymbol{u} \cdot \nabla_x p^{(0)}\, \mathrm{d}\Omega. \tag{4.5}$$

Consider formulation (4.5) successively with $v_i^{(0)} = k_{ip}(\boldsymbol{B}^{(0)})$ and $v_i^{(0)} = k_{iq}(-\boldsymbol{B}^{(0)})$, $V^{(1)} = -\eta_p(\boldsymbol{B}^{(0)})$ and $V^{(1)} = -\eta_q(-\boldsymbol{B}^{(0)})$, $u_i = k_{iq}(-\boldsymbol{B}^{(0)})$ and $u_i = k_{ip}(\boldsymbol{B}^{(0)})$, $\alpha = -\eta_q(-\boldsymbol{B}^{(0)})$ and $\alpha = -\eta_p(\boldsymbol{B}^{(0)})$, respectively. We obtain, after noting that permeability $\boldsymbol{K}$ does not depend on space variable $\boldsymbol{y}$,

$$\int_{\Omega} \sigma \left(\frac{\partial \eta_p(\boldsymbol{B}^{(0)})}{\partial y_i} + \epsilon_{ijk}\, k_{jp}(\boldsymbol{B}^{(0)}) B_k^{(0)}\right)\left(-\frac{\partial \eta_q(-\boldsymbol{B}^{(0)})}{\partial y_i} + \epsilon_{ijk}\, k_{jq}(-\boldsymbol{B}^{(0)})\, B_k^{(0)}\right) \mathrm{d}\Omega$$
$$+(k_{ip}(\boldsymbol{B}^{(0)}), k_{iq}(-\boldsymbol{B}^{(0)}))_{\mathscr{W}} = \Omega K_{pq}(-\boldsymbol{B}^{(0)}) = \Omega K_{pq}(-\langle \boldsymbol{B}^{(0)} \rangle),$$

$$\int_{\Omega} \sigma \left(\frac{\partial \eta_q(-\boldsymbol{B}^{(0)})}{\partial y_i} - \epsilon_{ijk}\, k_{jq}(-\boldsymbol{B}^{(0)})\, B_k^{(0)}\right)\left(-\frac{\partial \eta_p(\boldsymbol{B}^{(0)})}{\partial y_i} - \epsilon_{ijk}\, k_{jp}(\boldsymbol{B}^{(0)}) B_k^{(0)}\right) \mathrm{d}\Omega$$
$$+(k_{iq}(-\boldsymbol{B}^{(0)}), k_{ip}(\boldsymbol{B}^{(0)}))_{\mathscr{W}} = \Omega K_{qp}(\boldsymbol{B}^{(0)}) = \Omega K_{qp}(\langle \boldsymbol{B}^{(0)} \rangle).$$

Subtracting term by term these two equations yields

$$K_{pq}(-\langle \boldsymbol{B}^{(0)} \rangle) = K_{qp}(\langle \boldsymbol{B}^{(0)} \rangle). \tag{4.6}$$

This relation is the filtration analogue of the Hall's effect (see Landau & Lifshitz 1960, p. 96). It remains valid when Hall's effect is considered at the pore scale. However, when the tensor $\boldsymbol{K}$ is symmetric, see equation (4.4), the Hall constant cancels out and the tensor $\boldsymbol{K}$ appears as an even function of the magnetic field.

4.2. *Properties of the conductivity tensor* $\boldsymbol{\sigma}^{\mathrm{eff}}$

To investigate the properties of tensor $\boldsymbol{\sigma}^{\mathrm{eff}}$ we put $\nabla_x p^{(0)} = \mathbf{0}$ in formulations (3.18) and (3.19). Therefore relation (4.3) is now written

$$\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{u})_{\mathscr{W}} + \frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot (-\nabla_y \alpha + \boldsymbol{u} \times \boldsymbol{B}^{(0)})\, \mathrm{d}\Omega = 0. \tag{4.7}$$

By adding to the two terms of (4.7) the volume average over Ω of $-\boldsymbol{J}^{(0)} \cdot \nabla_x V^{(0)}$, we obtain

$$\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{u})_{\mathscr{W}} + \frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot (-\nabla_y \alpha - \nabla_x V^{(0)} + \boldsymbol{u} \times \boldsymbol{B}^{(0)})\, \mathrm{d}\Omega$$
$$= -\frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot \nabla_x V^{(0)}\, \mathrm{d}\Omega. \tag{4.8}$$

4.2.1. $\boldsymbol{\sigma}^{\text{eff}}$ *positive*

Putting now $\alpha = V^{(1)} \neq 0$ and $\boldsymbol{u} = \boldsymbol{v}^{(0)} \neq \boldsymbol{0}$ in formulation (4.8) yields

$$\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{v}^{(0)})_{\mathscr{W}} + \frac{1}{\Omega}\int_{\Omega} \frac{(\boldsymbol{J}^{(0)})^2}{\sigma}\,\mathrm{d}\Omega = -\frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot \nabla_x V^{(0)}\,\mathrm{d}\Omega = \boldsymbol{\sigma}^{\text{eff}} \nabla_x V^{(0)} \cdot \nabla_x V^{(0)} > 0,$$

which shows that the tensor $\boldsymbol{\sigma}^{\text{eff}}$ is positive.

4.2.2. $\boldsymbol{\sigma}^{\text{eff}}$ *symmetric*

In this section, we investigate the symmetries of $\boldsymbol{\sigma}^{\text{eff}}$. Quantities $v_i^{(0)} = \chi_{ip}$ and $V^{(1)} = -\tau_p$ are the solution of (3.18) and (3.19) for $\nabla_x p^{(0)} = \boldsymbol{0}$ and $\partial V^{(0)}/\partial x_i = -\delta_{ip}$. Consider formulation (4.8) successively with $v_i^{(0)} = \chi_{ip}$ and $v_i^{(0)} = \chi_{iq}$, $V^{(1)} = -\tau_p$ and $V^{(1)} = -\tau_q$, $u_i = \chi_{iq}$ and $u_i = \chi_{ip}$, $\alpha = -\tau_q$ and $\alpha = -\tau_p$, respectively. We obtain

$$(\chi_{ip}, \chi_{iq})_{\mathscr{W}} + \int_{\Omega} \sigma \left(\frac{\partial \tau_p}{\partial y_i} + \delta_{ip} + \epsilon_{ijk}\, \chi_{jp} B_k^{(0)} \right) \left(\frac{\partial \tau_q}{\partial y_i} + \delta_{iq} + \epsilon_{ijk}\, \chi_{jq}\, B_k^{(0)} \right) \mathrm{d}\Omega = \int_{\Omega_f} \sigma_{qp}\,\mathrm{d}\Omega = \Omega \sigma_{qp}^{\text{eff}}.$$

$$(\chi_{iq}, \chi_{ip})_{\mathscr{W}} + \int_{\Omega} \sigma \left(\frac{\partial \tau_q}{\partial y_i} + \delta_{iq} + \epsilon_{ijk}\, \chi_{jq} B_k^{(0)} \right) \left(\frac{\partial \tau_p}{\partial y_i} + \delta_{ip} + \epsilon_{ijk}\, \chi_{jp} B_k^{(0)} \right) \mathrm{d}\Omega = \int_{\Omega_f} \sigma_{pq}\,\mathrm{d}\Omega = \Omega \sigma_{pq}^{\text{eff}}.$$

Subtracting term by term these two equations yields

$$\sigma_{pq}^{\text{eff}} = \sigma_{qp}^{\text{eff}}, \tag{4.9}$$

which shows that $\boldsymbol{\sigma}^{\text{eff}}$ is symmetric. However, as for the permeability $\boldsymbol{K}$, assuming Hall's effect at the pore scale, yields $\sigma_{pq}^{\text{eff}} \neq \sigma_{qp}^{\text{eff}}$.

4.2.3. *Hall's effect*

By using (4.1), and after subtracting from the two terms of (4.7) the volume average over Ω of $\boldsymbol{J}^{(0)} \cdot \nabla_x V^{(0)}$, we obtain

$$\frac{1}{\Omega}(\boldsymbol{v}^{(0)}, \boldsymbol{u})_{\mathscr{W}} + \frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot (\nabla_y \alpha + \nabla_x V^{(0)} + \boldsymbol{u} \times \boldsymbol{B}^{(0)})\,\mathrm{d}\Omega = +\frac{1}{\Omega}\int_{\Omega} \boldsymbol{J}^{(0)} \cdot \nabla_x V^{(0)}\,\mathrm{d}\Omega. \tag{4.10}$$

Consider formulation (4.10) successively with $v_i^{(0)} = \chi_{ip}(\boldsymbol{B}^{(0)})$ and $v_i^{(0)} = \chi_{iq}(-\boldsymbol{B}^{(0)})$, $V^{(1)} = -\tau_p(\boldsymbol{B}^{(0)})$ and $V^{(1)} = -\tau_q(-\boldsymbol{B}^{(0)})$, $u_i = \chi_{iq}(-\boldsymbol{B}^{(0)})$ and $u_i = \chi_{ip}(\boldsymbol{B}^{(0)})$, $\alpha = -\tau_q(-\boldsymbol{B}^{(0)})$ and $\alpha = -\tau_p(\boldsymbol{B}^{(0)})$, respectively. We obtain, after noting that $\boldsymbol{\sigma}^{\text{eff}}$ does not depend on space variable $\boldsymbol{y}$,

$$\int_{\Omega} \sigma \left(\frac{\partial \tau_p(\boldsymbol{B}^{(0)})}{\partial y_i} + \delta_{ip} + \epsilon_{ijk}\, \chi_{jp}(\boldsymbol{B}^{(0)}) B_k^{(0)} \right) \times \left(-\frac{\partial \tau_q(-\boldsymbol{B}^{(0)})}{\partial y_i} - \delta_{iq} + \epsilon_{ijk}\, \chi_{jq}(-\boldsymbol{B}^{(0)}) B_k^{(0)} \right) \mathrm{d}\Omega + (\chi_{ip}(\boldsymbol{B}^{(0)}), \chi_{iq}(-\boldsymbol{B}^{(0)}))_{\mathscr{W}} = \Omega\, \sigma_{qp}^{\text{eff}}(\boldsymbol{B}^{(0)}) = \Omega\, \sigma_{qp}^{\text{eff}}(\langle \boldsymbol{B}^{(0)} \rangle),$$

$$\int_{\Omega} \sigma \left(\frac{\partial \tau_q(-\boldsymbol{B}^{(0)})}{\partial y_i} + \delta_{iq} - \epsilon_{ijk}\, \chi_{jq}(-\boldsymbol{B}^{(0)}) B_k^{(0)} \right) \times \left(-\frac{\partial \tau_p(\boldsymbol{B}^{(0)})}{\partial y_i} - \delta_{ip} - \epsilon_{ijk}\, \chi_{jp}(\boldsymbol{B}^{(0)}) B_k^{(0)} \right) \mathrm{d}\Omega + (\chi_{iq}(-\boldsymbol{B}^{(0)}), \chi_{ip}(\boldsymbol{B}^{(0)}))_{\mathscr{W}} = \Omega \sigma_{pq}^{\mathrm{eff}}(-\boldsymbol{B}^{(0)}) = \Omega \sigma_{pq}^{\mathrm{eff}}(-\langle \boldsymbol{B}^{(0)} \rangle).$$

Subtracting term by term these two equations yields

$$\sigma_{pq}^{\mathrm{eff}}(-\langle \boldsymbol{B}^{(0)} \rangle) = \sigma_{qp}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle). \tag{4.11}$$

This relation represents the Hall effect (see Landau & Lifshitz 1960, p. 96). This relation remains valid when Hall's effect is considered at the pore scale. However, when $\boldsymbol{\sigma}^{\mathrm{eff}}$ is symmetric, see equation (4.9), the Hall constant cancels out and $\boldsymbol{\sigma}^{\mathrm{eff}}$ appears as an even function of the magnetic field.

4.3. *Properties of* $\boldsymbol{\chi}^{\mathrm{eff}}$ *and* $\boldsymbol{\eta}^{\mathrm{eff}}$

4.3.1. *Relation between* $\boldsymbol{\chi}^{\mathrm{eff}}$ *and* $\boldsymbol{\eta}^{\mathrm{eff}}$

Consider formulation (4.3) with $\nabla_x V^{(0)} = \mathbf{0}$, $\partial p^{(0)}/\partial x_i = -\delta_{iq}$, $v_i^{(0)} = k_{iq}$, $V^{(1)} = -\eta_q$, $u_i^{(0)} = \chi_{ip}$, $\alpha = -\tau_p$ and formulation (4.7) with $\nabla_x p^{(0)} = \mathbf{0}$, $\partial V^{(0)}/\partial x_i = -\delta_{ip}$, $v_i^{(0)} = \chi_{ip}$, $V^{(1)} = -\tau_p$, $u_i^{(0)} = k_{iq}$, $\alpha = -\eta_q$. We obtain successively

$$(k_{iq}, \chi_{ip})_{\mathscr{W}} + \int_{\Omega} \sigma \left(\frac{\partial \eta_q}{\partial y_i} + \epsilon_{ijk}\, k_{jq}\, B_k^{(0)} \right) \left(\frac{\partial \tau_p}{\partial y_i} + \epsilon_{ijk}\, \chi_{ip}\, B_k^{(0)} \right) \mathrm{d}\Omega = \int_{\Omega_f} \chi_{qp}\, \mathrm{d}\Omega = \Omega \chi_{qp}^{\mathrm{eff}},$$

$$(\chi_{ip}, k_{iq})_{\mathscr{W}} + \int_{\Omega} \sigma \left(\frac{\partial \tau_p}{\partial y_i} + \delta_{ip} + \epsilon_{ijk}\, \chi_{jp}\, B_k^{(0)} \right) \left(\frac{\partial \eta_q}{\partial y_i} + \epsilon_{ijk}\, k_{jq}\, B_k^{(0)} \right) \mathrm{d}\Omega = 0.$$

Subtracting term by term gives

$$\int_{\Omega} \sigma \left(-\frac{\partial \eta_q}{\partial y_p} - \epsilon_{pjk}\, k_{jq}\, B_k^{(0)} \right) \mathrm{d}\Omega = \Omega\, \chi_{qp}^{\mathrm{eff}},$$

which, when considering (3.28), can be written in the form

$$\eta_{pq}^{\mathrm{eff}} = -\chi_{qp}^{\mathrm{eff}}. \tag{4.12}$$

Onsager's relation for the coupling between the macroscopic mass flux and electric current is satisfied (de Groot & Mazur 1969). Note however that, unlike Onsager's approach, the symmetry of the coupling is obtained here from the local field properties by using a deterministic approach.

4.3.2. $\boldsymbol{\chi}^{eff}$ *and* $\boldsymbol{\eta}^{eff}$ *odd functions of the magnetic field*

Consider formulation (3.18) successively with $V^{(1)} = V^{(1)}(\boldsymbol{B}^{(0)})$ and $V^{(1)} = V^{(1)}(-\boldsymbol{B}^{(0)})$ and subtract term by term. We obtain

$$\forall\, \alpha \in \mathscr{V}, \quad (V^{(1)}(\boldsymbol{B}^{(0)}) - V^{(1)}(-\boldsymbol{B}^{(0)}), \alpha)_{\mathscr{V}} = \int_{\Omega} \sigma [(\boldsymbol{v}^{(0)}(\boldsymbol{B}^{(0)}) + \boldsymbol{v}^{(0)}(-\boldsymbol{B}^{(0)})) \times \boldsymbol{B}^{(0)}] \cdot \nabla_y \alpha \, \mathrm{d}\Omega. \tag{4.13}$$

Consider now formulation (3.19) successively with $\boldsymbol{v}^{(0)} = \boldsymbol{v}^{(0)}(\boldsymbol{B}^{(0)})$ and $\boldsymbol{v}^{(0)} = \boldsymbol{v}^{(0)}(-\boldsymbol{B}^{(0)})$ and add term by term. We have

$$\forall \boldsymbol{u} \in \mathscr{W}, \quad (\boldsymbol{v}(\boldsymbol{B}^{(0)}) + \boldsymbol{v}^{(0)}(-\boldsymbol{B}^{(0)}), \boldsymbol{u})_{\mathscr{W}} = -2 \int_{\Omega_f} \boldsymbol{u} \cdot \nabla_x p^{(0)} \, \mathrm{d}\Omega$$
$$+ \int_{\Omega_f} \sigma_f [((\boldsymbol{v}^{(0)}(\boldsymbol{B}^{(0)}) + \boldsymbol{v}^{(0)}(-\boldsymbol{B}^{(0)})) \times \boldsymbol{B}^{(0)}) \times \boldsymbol{B}^{(0)}] \cdot \boldsymbol{u} \, \mathrm{d}\Omega$$
$$- \int_{\Omega_f} \sigma_f [(\nabla_y V^{(1)}(\boldsymbol{B}^{(0)}) - \nabla_y V^{(1)}(-\boldsymbol{B}^{(0)})) \times \boldsymbol{B}^{(0)}] \cdot \boldsymbol{u} \, \mathrm{d}\Omega. \tag{4.14}$$

Formulations (4.13) and (4.14) ensure the existence of solutions for $V^{(1)}(\boldsymbol{B}^{(0)}) - V^{(1)}(-\boldsymbol{B}^{(0)})$ and $\boldsymbol{v}^{(0)}(\boldsymbol{B}^{(0)}) + \boldsymbol{v}^{(0)}(-\boldsymbol{B}^{(0)})$ that are linear vectorial functions of $\nabla_x p^{(0)}$. In particular we have

$$\boldsymbol{v}^{(0)}(\boldsymbol{B}^{(0)}) + \boldsymbol{v}^{(0)}(-\boldsymbol{B}^{(0)}) = \gamma \, \nabla_x p^{(0)},$$

where $\boldsymbol{\gamma}$ is a tensor which depends on $\boldsymbol{y}$ and $\boldsymbol{B}^{(0)}$. On an other hand, relation (3.20) gives

$$\boldsymbol{v}^{(0)}(\boldsymbol{B}^{(0)}) + \boldsymbol{v}^{(0)}(-\boldsymbol{B}^{(0)}) = -(\boldsymbol{k}(\boldsymbol{B}^{(0)}) + \boldsymbol{k}(-\boldsymbol{B}^{(0)}))\nabla_x p^{(0)} - (\boldsymbol{\chi}(\boldsymbol{B}^{(0)}) + \boldsymbol{\chi}(-\boldsymbol{B}^{(0)}))\nabla_x V^{(0)}.$$

Identification of the above two relations yields

$$\boldsymbol{\chi}(\boldsymbol{B}^{(0)}) + \boldsymbol{\chi}(-\boldsymbol{B}^{(0)}) = 0,$$

from which we deduce

$$\boldsymbol{\chi}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) = -\boldsymbol{\chi}^{\mathrm{eff}}(-\langle \boldsymbol{B}^{(0)} \rangle) \tag{4.15}$$

Due to (4.12) there is a similar relation for $\boldsymbol{\eta}^{\mathrm{eff}}$.

4.4. *Isotropic porous media*

Consider now a macroscopically isotropic porous medium of permeability $K(0)$ and of effective conductivity $\sigma^{\mathrm{eff}}(0)$ in the absence of a magnetic field. Obviously, $\eta^{\mathrm{eff}}(0) = 0$ and $\chi^{\mathrm{eff}}(0) = 0$. Apply a magnetic field, e.g. in the direction X_3. All the effective tensors are $\langle \boldsymbol{B}^{(0)} \rangle$-dependent. However, they should be invariant to rotations around the $\boldsymbol{e}_3$-axis. Tensor $\boldsymbol{K}(\langle \boldsymbol{B}^{(0)} \rangle)$ is a symmetric tensor (4.4). Its invariance under rotations of axis $\boldsymbol{e}_3$ imposes that its non-diagonal components cancel out. Due to isotropy, diagonal components in the plane $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ are equal. There is a similar form for tensor $\boldsymbol{\sigma}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle)$. We have

$$\boldsymbol{K}(\langle \boldsymbol{B}^{(0)} \rangle) = \begin{pmatrix} K(\langle \boldsymbol{B}^{(0)} \rangle) & 0 & 0 \\ 0 & K(\langle \boldsymbol{B}^{(0)} \rangle) & 0 \\ 0 & 0 & K(0) \end{pmatrix},$$

$$\boldsymbol{\sigma}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) = \begin{pmatrix} \sigma^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) & 0 & 0 \\ 0 & \sigma^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) & 0 \\ 0 & 0 & \sigma^{\mathrm{eff}}(0) \end{pmatrix}.$$

Consider now the electro-osmotic conductivity $\boldsymbol{\chi}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle)$. Change the axes $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ to $(\boldsymbol{e}_2, \boldsymbol{e}_1, -\boldsymbol{e}_3)$. Isotropy imposes $\chi_{11}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) = \chi_{22}^{\mathrm{eff}}(-\langle \boldsymbol{B}^{(0)} \rangle)$. Isotropy also implies that diagonal components in the plane $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ are equal. By using relation (4.15) we obtain

$$\chi_{11}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) = \chi_{22}^{\mathrm{eff}}(-\langle \boldsymbol{B}^{(0)} \rangle) = \chi_{11}^{\mathrm{eff}}(-\langle \boldsymbol{B}^{(0)} \rangle) = -\chi_{11}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle).$$

Diagonal components cancel out. Finally, the invariance under rotations of axis $\boldsymbol{e}_3$

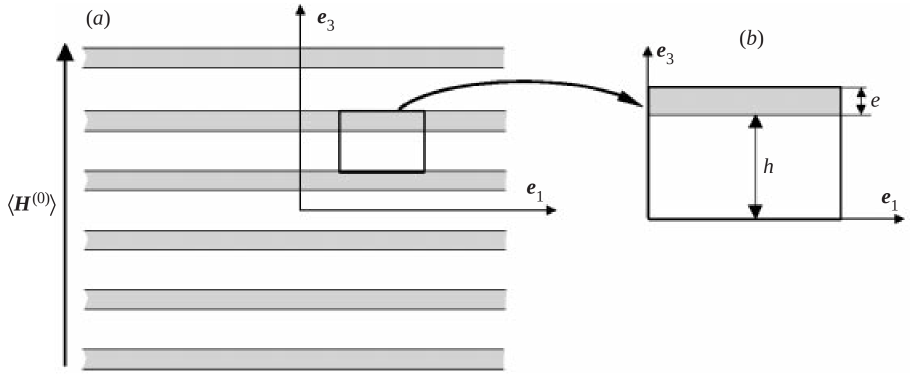

FIGURE 2. (a) Macroscopic porous medium. (b) Periodic cell Ω.

imposes that tensor $\boldsymbol{\chi}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle)$ (and in the same way tensor $\boldsymbol{\eta}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle)$ is antisymmetric in the plane $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. With relation (4.12), we have

$$\boldsymbol{\chi}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) = \begin{pmatrix} 0 & \chi^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) & 0 \\ -\chi^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boldsymbol{\eta}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle).$$

In the case of isotropy, the electric conductivity $\boldsymbol{\eta}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle)$ associated with the streaming potential effect is equal to the electro-osmotic conductivity $\boldsymbol{\chi}^{\mathrm{eff}}(\langle \boldsymbol{B}^{(0)} \rangle)$.

5. Example

Analytical results are of great interest because they permit us to point out general features concerning effective coefficients. Unfortunately, such results are available for only a few pore geometries. We investigate in this section porous media in which the pore system consists of parallel plane fissures, as shown in figure 2. The porosity is ϕ and the pore thickness is denoted h. We denote $\boldsymbol{x} = (x_1, x_2, x_3)$ the macroscopic dimensional space variable and $\boldsymbol{y} = (y_1, y_2, y_3)$ the local dimensional space variable. The conducting porous medium is rigid. Both solid and saturating fluid are subjected to a constant macroscopic magnetic field $\langle \boldsymbol{H}^{(0)} \rangle = \langle H_3^{(0)} \rangle \boldsymbol{e}_3$, perpendicular to the pore surfaces and to a macroscopic electric potential $V^{(0)}(\boldsymbol{x})$. The fluid is also subjected to a macroscopic gradient of pressure $(\mathrm{d}p^{(0)}/\mathrm{d}x_1)\,\boldsymbol{e}_1 + (\mathrm{d}p^{(0)}/\mathrm{d}x_2)\,\boldsymbol{e}_2$. Consequently, the problem to be solved is reduced to the well-known Hartmann problem (Hartmann 1937). The pore domain under consideration is $0 \leqslant y_3 \leqslant (h + e)$. The periodicity is arbitrary in the directions y_1 and y_2. Therefore, p, $\boldsymbol{v}$, $\boldsymbol{B}$, $\boldsymbol{H}$, ψ, $\boldsymbol{E}$, V and $\boldsymbol{J}$ are functions of y_3, only.

5.1. *Macroscopic magnetic field and induction*

The macroscopic field satisfies equations (3.5) and (3.6),

$$\langle \boldsymbol{H}^{(0)} \rangle = \langle H_3^{(0)} \rangle \boldsymbol{e}_3 = -\nabla_x \psi^{(0)},$$
$$\nabla_x \times \langle \boldsymbol{H}^{(0)} \rangle = \boldsymbol{0}.$$

Therefore, we have $\psi^{(0)} = \psi^{(0)}(x_3)$ and

$$H_1^{(0)} = 0, \quad H_2^{(0)} = 0, \quad H_3^{(0)} = -\frac{\mathrm{d}\psi^{(1)}}{\mathrm{d}y_3} - \frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}x_3}.$$

Equations (3.2) and (3.3) take the form

$$\frac{\mathrm{d}}{\mathrm{d}y_3}\left[\mu^*\left(\frac{\mathrm{d}\psi^{(1)}}{\mathrm{d}y_3}+\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}x_3}\right)\right]=0 \quad \text{in } \Omega,$$

$$\mu_f^*\left(\frac{\mathrm{d}\psi_f^{(1)}}{\mathrm{d}y_3}+\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}x_3}\right)=\mu_s^*\left(\frac{\mathrm{d}\psi_s^{(1)}}{\mathrm{d}y_3}+\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}x_3}\right) \quad \text{on } \Gamma \,(\text{at } y_3=0,h),$$

where $\mathrm{d}\psi^{(0)}/\mathrm{d}x_3$ is a constant. The unknown $\psi^{(1)}(x_3,y_3)$ is Ω-periodic and continuous on Γ. In this particular case, the general solution (3.4) for the potential $\psi^{(1)}$ reduces to

$$\psi_s^{(1)}(x_3,y_3)=\frac{h\,(\mu_f^*-\mu_s^*)}{h\mu_s^*+e\mu_f^*}(y_3-(h+e))\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}x_3}+\bar{\psi}^{(1)}(x_3) \quad \text{in } \Omega_s,$$

$$\psi_f^{(1)}(x_3,y_3)=\frac{e(\mu_s^*-\mu_f^*)}{h\mu_s^*+e\mu_f^*}y_3\,\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}x_3}+\bar{\psi}^{(1)}(x_3) \quad \text{in } \Omega_f.$$

Therefore, from equation (3.10), we obtain

$$B_1^{(0)}=0, \quad B_2^{(0)}=0, \quad B_3^{(0)}=-\frac{(h+e)\mu_f^*\mu_s^*}{h\mu_s^*+e\mu_f^*}\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}x_3}=\frac{(h+e)\mu_f^*\mu_s^*}{h\mu_s^*+e\mu_f^*}\langle H_3^{(0)}\rangle \quad \text{in} \quad \Omega.$$

The magnetic induction is independent of the local dimensional variable $\boldsymbol{y}$. Finally, the macroscopic model for the magnetic induction (3.9) takes the form,

$$\nabla_x \cdot \langle \boldsymbol{B}^{(0)}\rangle=0, \quad \langle \boldsymbol{B}^{(0)}\rangle=\langle B_3^{(0)}\rangle \boldsymbol{e}_3,$$

where

$$\langle B_3^{(0)}\rangle=\mu_{33}^{*\mathrm{eff}}\langle H_3^{(0)}\rangle, \quad \mu_{33}^{*\mathrm{eff}}=\frac{\mu_f^*\mu_s^*}{\phi\mu_s^*+(1-\phi)\mu_f^*}.$$

5.2. *Macroscopic mass and electric fluxes*

At the lowest order, the local electric field $\boldsymbol{E}^{(0)}$ is in the form

$$E_1^{(0)}=-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_1}, \quad E_2^{(0)}-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_2}, \quad E_3^{(0)}=-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_3}-\frac{\mathrm{d}V^{(1)}}{\mathrm{d}y_3}.$$

The set of equations (3.13)–(3.17) becomes

$$\mu\,\frac{\mathrm{d}^2 v_1^{(0)}}{\mathrm{d}y_3^2}-\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_1}+\sigma_f B_3^{(0)}\left(-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_2}-B_3^{(0)}v_1^{(0)}\right)=0 \quad \text{in } \Omega_f, \tag{5.1}$$

$$\mu\frac{\mathrm{d}^2 v_2^{(0)}}{\mathrm{d}y_3^2}-\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_2}-\sigma_f\;B_3^{(0)}\left(-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_1}+B_3^{(0)}v_2^{(0)}\right)=0 \quad \text{in } \Omega_f, \tag{5.2}$$

$$\mu\,\frac{\mathrm{d}^2 v_3^{(0)}}{\mathrm{d}y_3^2}-\frac{\mathrm{d}p^{(1)}}{\mathrm{d}y_3}=0 \quad \text{in } \Omega_f, \tag{5.3}$$

$$\frac{\mathrm{d}v_3^{(0)}}{\mathrm{d}y_3}=0, \quad \text{in } \Omega_f, \tag{5.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}y_3}\left[\sigma\left(-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_3}-\frac{\mathrm{d}V^{(1)}}{\mathrm{d}y_3}\right)\right]=0 \quad \text{in } \Omega, \tag{5.5}$$

$$v_1^{(0)}=v_2^{(0)}=v_3^{(0)}=0 \quad \text{on } \Gamma \,(\text{at } y_3=0,h), \tag{5.6}$$

$$\sigma_f\left(-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_3}-\frac{\mathrm{d}V_f^{(1)}}{\mathrm{d}y_3}\right)=\sigma_s\left(-\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_3}-\frac{\mathrm{d}V_s^{(1)}}{\mathrm{d}y_3}\right)\quad \text{on } \Gamma \text{ (at } y_3=0,h), \tag{5.7}$$

where $\mathrm{d}V^{(0)}/\mathrm{d}x_1$, $\mathrm{d}V^{(0)}/\mathrm{d}x_2$, $\mathrm{d}p^{(0)}/\mathrm{d}x_1$, $\mathrm{d}p^{(0)}/\mathrm{d}x_2$ and $B_3^{(0)}$ are constants. The unknowns $V^{(1)}$, $\boldsymbol{v}^{(0)}$ and $p^{(1)}$ are Ω-periodic and the potential $V^{(1)}$ is also continuous on Γ. The above set of differential equations of variable y_3 is easily solved. It can be shown that the unknowns $V^{(1)}$, $\boldsymbol{v}^{(0)}$ and $p^{(1)}$ given by equations (3.20)–(3.21) can be put into the form

$$V_s^{(1)}(\boldsymbol{x},y_3)=\frac{h(\sigma_f-\sigma_s)}{h\sigma_s+e\sigma_f}(y_3-(h+e))\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_3}+\bar{V}^{(1)}(\boldsymbol{x})\quad \text{in } \Omega_s,$$

$$V_f^{(1)}(\boldsymbol{x},y_3)=\frac{e(\sigma_s-\sigma_f)}{h\sigma_s+e\sigma_f}y_3\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_3}+\bar{V}^{(1)}(\boldsymbol{x})\quad \text{in } \Omega_f,$$

$$v_1^{(0)}=\frac{h^2}{4Ha^2}\left(-\frac{1}{\mu}\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_1}-\frac{2Ha}{h}\left(\frac{\sigma_f}{\mu}\right)^{1/2}\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_2}\right)\left(1-\frac{\cosh(2Ha\ y_3/h)}{\cosh Ha}\right),$$

$$v_2^{(0)}=\frac{h^2}{4Ha^2}\left(-\frac{1}{\mu}\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_2}+\frac{2Ha}{h}\left(\frac{\sigma_f}{\mu}\right)^{1/2}\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_1}\right)\left(1-\frac{\cosh(2Ha\ y_3/h)}{\cosh Ha}\right),$$

$$v_3^{(0)}=0,$$

$$p^{(1)}=\overline{p}^{(1)}(\boldsymbol{x}),\quad \boldsymbol{\zeta}=\boldsymbol{\xi}=\boldsymbol{0},$$

where we have used $h/2$ as the characteristic length and the Hartmann number is defined by

$$Ha=\left(\frac{\sigma}{\mu}\right)^{1/2}B_3^{(0)}\frac{h}{2}.$$

By averaging we obtain the macroscopic flux (3.25) in the form

$$\langle v_1^{(0)}\rangle=-K\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_1}-\chi^{\mathrm{eff}}\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_2},\quad \langle v_2^{(0)}\rangle=-K\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_2}+\chi^{\mathrm{eff}}\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_1},$$

where

$$K=K_{11}=K_{22}=\frac{\phi h^2}{4\mu Ha^2}\left(1-\frac{\tanh Ha}{Ha}\right),$$

$$\chi^{\mathrm{eff}}=\chi_{12}^{\mathrm{eff}}=-\chi_{21}^{\mathrm{eff}}=\frac{\phi h}{2Ha}\left(\frac{\sigma_f}{\mu}\right)^{1/2}\left(1-\frac{\tanh Ha}{Ha}\right).$$

Clearly, $K_{12}=K_{21}=0$ and $\chi_{11}^{\mathrm{eff}}=\chi_{22}^{\mathrm{eff}}=0$. Tensor $\boldsymbol{\chi}^{\mathrm{eff}}$ is antisymmetric. The permeability K is strongly affected by the magnetic flux. As $B_3^{(0)}=0$, i.e. $\langle H_3^{(0)}\rangle=0$, we recover the permeability in absence of magnetic flux

$$K(0)=\frac{\phi h^2}{12\mu}.$$

Finally, we obtain the macroscopic current density (3.27), which is the volume average of the local current density over $\Omega=\Omega_s+\Omega_f$:

$$\langle J_1^{(0)}\rangle=-\sigma^{\mathrm{eff}}\frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_1}-\eta^{\mathrm{eff}}\frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_2},$$

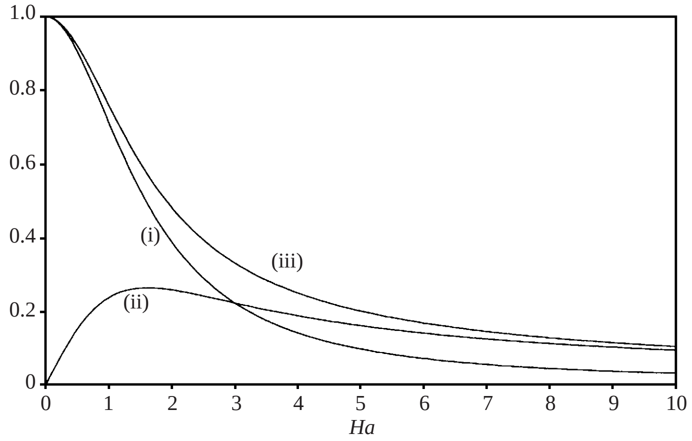

FIGURE 3. Dimensionless macroscopic coefficients versus Hartmann number. (i) Dimensionless permeability $K_d(Ha)$. (ii) $\chi_d^{\text{eff}}(Ha) = \eta_d^{\text{eff}}(Ha)$. (iii) Dimensionless macroscopic conductivity $\sigma_d^{\text{eff}}(Ha)$ with $\sigma_s = 0$.

$$\langle J_2^{(0)} \rangle = -\sigma^{\text{eff}} \frac{\mathrm{d}V^{(0)}}{\mathrm{d}x_2} + \eta^{\text{eff}} \frac{\mathrm{d}p^{(0)}}{\mathrm{d}x_1},$$
$$\langle J_3^{(0)} \rangle = 0,$$

where

$$\sigma^{\text{eff}} = \sigma_{11}^{\text{eff}} = \sigma_{22}^{\text{eff}} = (1-\phi)\sigma_s + \phi\sigma_f - \sigma_f B_3^{(0)} \chi^{\text{eff}},$$
$$\eta^{\text{eff}} = \eta_{12}^{\text{eff}} = -\eta_{21}^{\text{eff}} = \sigma_f B_3^{(0)} K = \chi^{\text{eff}},$$

and we have

$$\sigma_{12}^{\text{eff}} = \sigma_{21}^{\text{eff}} = \eta_{11}^{\text{eff}} = \eta_{22}^{\text{eff}} = 0.$$

Note the Onsager relation $\eta^{\text{eff}} = \chi^{\text{eff}}$ that represents (4.12) and characterizes the coupling between the macroscopic mass flux and electric current. The evolution with Hartmann number of dimensionless numbers

$$K_d(Ha) = \frac{K}{K(0)} = \frac{3}{Ha^2}\left(1 - \frac{\tanh Ha}{Ha}\right),$$
$$\chi_d^{\text{eff}}(Ha) = \eta_d^{\text{eff}}(Ha) = \frac{\chi^{\text{eff}}}{(\sigma_f/\mu)^{1/2}\,\phi\, h/2} = \frac{1}{Ha}\left(1 - \frac{\tanh Ha}{Ha}\right),$$
$$\sigma_d^{\text{eff}}(Ha) = \frac{\sigma^{\text{eff}}(\sigma_s = 0)}{\sigma_f\,\phi} = \frac{\tanh Ha}{Ha},$$

is shown in figure 3, see curves (i), (ii) and (iii) respectively.

This simple example illustrates that some features of MHD in homogeneous fluids remain valid at a macroscopic scale in porous media: due to the body force $\boldsymbol{E} \times \boldsymbol{B}$ exerted at the pore level, a macroscopic electric field $\mathrm{d}V^{(0)}/\mathrm{d}x_2$ in direction y_2 causes a macroscopic mass flow in direction y_1; due to the electromotive field $\boldsymbol{v} \times \boldsymbol{B}$ at the pore level, a macroscopic gradient of pressure in direction y_1 causes a macroscopic electric current in direction y_2. These results remain valid for more intricate pore geometry.

6. Concluding remarks

We have investigated the macroscopic description of the seepage of conductive fluids in porous media in the presence of a magnetic field, for finite Hartmann number, i.e. $\varepsilon \ll Ha \ll \varepsilon^{-1}$ and finite load factor, i.e. $\varepsilon \ll \mathscr{K} \ll \varepsilon^{-1}$, where ε characterizes the separation of scales. Returning to *dimensional* quantities, the equivalent macroscopic description is given by

$$\nabla \times \langle \boldsymbol{H} \rangle = O(\varepsilon) \quad \text{with} \quad \langle \boldsymbol{H} \rangle = -\nabla \psi + O(\varepsilon), \tag{6.1}$$

$$\nabla \cdot \langle \boldsymbol{B} \rangle = O(\varepsilon) \quad \text{with} \quad \langle \boldsymbol{B} \rangle = \boldsymbol{\mu}^{*\text{eff}} \langle \boldsymbol{H} \rangle + O(\varepsilon), \tag{6.2}$$

$$\nabla \times \langle \boldsymbol{E} \rangle = O(\varepsilon), \quad \text{with} \quad \langle \boldsymbol{E} \rangle = -\nabla V + O(\varepsilon), \tag{6.3}$$

$$\nabla \cdot \langle \boldsymbol{J} \rangle = O(\varepsilon) \quad \text{with} \quad \langle \boldsymbol{J} \rangle = -\boldsymbol{\sigma}^{\text{eff}} \nabla V - \boldsymbol{\eta}^{\text{eff}} \nabla p + O(\varepsilon), \tag{6.4}$$

$$\nabla \cdot \langle \boldsymbol{v} \rangle = O(\varepsilon) \quad \text{with} \quad \langle \boldsymbol{v} \rangle = -\boldsymbol{K} \nabla p - \boldsymbol{\chi}^{\text{eff}} \nabla V + O(\varepsilon), \tag{6.5}$$

where $\boldsymbol{K}$ is the permeability, $\boldsymbol{\sigma}^{\text{eff}}$ is the effective electric conductivity, $\boldsymbol{\chi}^{\text{eff}}$ is the electro-osmotic conductivity and $\boldsymbol{\eta}^{\text{eff}}$ is the electric conductivity associated with the streaming potential effect. Under typical conditions like those encountered in magnetohydrodynamic flows, the mass flow and the electric current are described by two coupled equations which are both linear relations of the macroscopic gradient of pressure and of electric potential. Solving macroscopic boundary value problems requires solving the two coupled balances (6.4) and (6.5). Due to the small value of the magnetic Reynolds number under consideration, the macroscopic magnetic field is described by an independent classical magnetic field equation.

As already noted from Hartmann (1937), the permeability $\boldsymbol{K}$ is strongly affected by the presence of a magnetic field. However, the permeability tensor remains positive, as for seepage flows in the absence of magnetic field. The effective conductivity tensor $\boldsymbol{\sigma}^{\text{eff}}$ which depends on the magnetic field is also positive and we have

$$K_{pq}(\langle \boldsymbol{B} \rangle) = K_{qp}(\langle \boldsymbol{B} \rangle), \quad \sigma^{\text{eff}}_{pq}(\langle \boldsymbol{B} \rangle) = \sigma^{\text{eff}}_{qp}(\langle \boldsymbol{B} \rangle). \tag{6.6}$$

Finally, we have shown by following a deterministic approach that the different effective coefficients satisfy classical Onsager relations (see Groot & Mazur 1969, p. 39). In particular, the permeability $\boldsymbol{K}$ and the effective conductivity $\boldsymbol{\sigma}^{\text{eff}}$ satisfy the filtration analogue of the Hall effect and the Hall effect, respectively (see Landau & Lifshitz 1960, p. 96),

$$K_{pq}(-\langle \boldsymbol{B} \rangle) = K_{qp}(\langle \boldsymbol{B} \rangle), \quad \sigma^{\text{eff}}_{pq}(-\langle \boldsymbol{B} \rangle) = \sigma^{\text{eff}}_{qp}(\langle \boldsymbol{B} \rangle). \tag{6.7}$$

Relations (6.6) and (6.7) show that $\boldsymbol{K}$ and $\boldsymbol{\sigma}^{\text{eff}}$ are even tensorial functions of the magnetic induction $\langle \boldsymbol{B} \rangle$. Moreover the coupling between the macroscopic mass flux and electric current is characterized by

$$\eta^{\text{eff}}_{pq} = -\chi^{\text{eff}}_{qp}, \quad \boldsymbol{\chi}^{\text{eff}}(\langle \boldsymbol{B} \rangle) = -\boldsymbol{\chi}^{\text{eff}}(-\langle \boldsymbol{B} \rangle), \quad \boldsymbol{\eta}^{\text{eff}}(\langle \boldsymbol{B} \rangle) = -\boldsymbol{\eta}^{\text{eff}}(-\langle \boldsymbol{B} \rangle) \tag{6.8}$$

Tensors $\boldsymbol{\eta}^{\text{eff}}$ and $\boldsymbol{\chi}^{\text{eff}}$ are odd tensorial functions of $\langle \boldsymbol{B} \rangle$.

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